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On prediction of the turbulent flow over a wavy boundary

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The importance of fluctuating turbulent stresses in the flow over a wave is examined. It is shown that anisotropic stresses, which are most likely to be turbulent Reynolds stresses, are essential to the process of energy flow to the wave. Two fundamentally different methods of predicting fluctuating turbulent Reynolds stresses are examined. One method makes use of a phenomenological closure of the conservation equation for the turbulent Reynolds stresses and is similar to the turbulent boundary-layer calculation scheme of Bradshaw, Ferriss & Atwell (1967). The second method is based on the assumption that the turbulent stresses are determined by the recent history of velocity shear experienced by a fluid parcel and results in a viscoelastic constitutive relation for the turbulence; in the limit of shortest 'memory' this relation becomes the eddy viscosity model proposed by Hussain & Reynolds (1970). Comparison of predicted and measured values of surface pressure indicates that the eddy viscoelasticity model can explain measured pressure distributions but the comparison is not conclusive. Suggestions for further measurements are made.

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1. Introduction

Since Miles (1957) proposed his quasi-laminar model of flow over a wave it has become apparent that a satisfactory description of this phenomenon must include the interaction between the wave and the turbulent components of the flow. Intuitively this is reasonable since it is the turbulent components of the flow which transfer momentum to a flat surface (except very near the surface) and it would therefore be surprising if they did not play a significant role in flow over a wavy surface. From a theoretical point of view the neglect of turbulent stresses and nonlinearities in the analysis of flow over a wave dictates that all the momentum transferred to the wave is extracted from the critical layer by the action of molecular viscosity (this is proved in the following section). If nonlinearities are included, then only the smallest waves can receive momentum through this mechanism (see Davis 1969) but it was found by Davis (1970) that even when the dynamics are linear small fluctuations of the turbulent Reynolds stresses seriously alter the wave-induced flow, indicating that the quasi-laminar model is very sensitive to the assumption that there are no fluctuations of the turbulent stresses.

The objections raised above are, however, all indirect, and without the experimental evidence now available the debate could only end inconclusively. Stewart's (1970) measurements of the mean flow over waves in a laboratory showed significant departures from the quasi-laminar model. Kendall's (1970) laboratory study of flow over a wavy wall not only showed discrepancies between measured and predicted mean-flow quantities but also demonstrated that the turbulence is affected by the wave and that the turbulent stresses are not constant. Finally, Dobson (1969) has shown that under field conditions the work done on waves by the mean normal stress is seriously underestimated by the quasi-laminar model. This evidence together with the theoretical objections seems sufficient to lead to the conclusion that the Miles quasi-laminar model is incorrect.

Miles (1967) discussed the problem of including wave-induced variations of turbulent Reynolds stress in the analysis of flow over a wave. This requires the introduction of some phenomenological model to describe the response of the turbulence to mean-flow fluctuations and then the application of this model to the flow over a wave. Miles concluded that such a step was not warranted by the then available experimental data. There is now sufficient data available to impose a fairly stringent test on any model and it is the purpose of this paper to report the results of some attempts to apply models of turbulence to the problem of flow over a wave.

The formulation of the problem of interest was first accomplished by Miles (1957) and has been discussed more recently by Phillips (1966) and Miles (1967). Consider a wavy boundary which, when viewed in a frame of reference translating at the wave speed c, is of the form

$$x_3\,(\text{surface}) = a\eta = a\cos x_1,$$

where all lengths have been scaled by the inverse wavenumber of the wave train. Define the mean of any quantity as the average value obtained at any particular choice of x_1 and x_3 and the overall mean as the average of the mean quantities at all x_1. Mean quantities (denoted with a capital and caret) may be considered as the sum of an overall mean part (denoted by a capital letter or by an overbar) and a wave-induced part (denoted by a capital script letter). The total quantity is then the sum of a mean value and a turbulent component (denoted by a lower case letter). Thus the components of the total velocity are $\hat{U}_n + u_n$, where all velocities are scaled by a characteristic velocity U_0. The mean velocity is

$$\hat{U}_n = \delta_{1n}\,U(x_3) + a\mathscr{U}_n,$$

where the delta with subscripts is the Kronecker delta and $\mathscr{U}_2 = 0$.

For the purposes of this paper, viscous stresses will usually be considered negligible so that the total mean stress tensor will be nearly the sum of the isotropic pressure component $\hat{P} = P + a\mathscr{P}$ and Reynolds stress

$$\hat{R}_{nm} = \text{mean}\,(-u_n u_m) = R_{nm} + a\mathscr{R}_{nm},$$

where all stresses are scaled by the product of density and U_0^2. The total stress tensor is then

$$\hat{S}_{nm} = -\delta_{nm}\hat{P} + \hat{R}_{nm} + \text{viscous stress} = S_{nm} + a\mathscr{S}_{nm}.$$

By definition the overall mean quantities are functions only of x_3 and consequently must obey the overall averaged momentum equations, which reduce to

$$\frac{d}{dx_3}S_{13} = \frac{d}{dx_3}S_{33} = 0.$$

The total stresses are constants and therefore the stresses R_{13} and $R_{33}-P$ are nearly constant except possibly very near the wave surface, where viscous effects may become important.

If the dimensionless wave amplitude is sufficiently small it is plausible that the dynamics of the perturbation quantities become linear and that terms of $O(a^2)$ can be neglected. This is a crucial assumption since nonlinear effects can alter the nature of the flow in a fundamental way (Davis 1969). If terms of $O(a^2)$ are neglected the mean momentum equation becomes

$$U\,\partial_1 \mathscr{U}_n + \delta_{n1}\, U'\mathscr{U}_3 = \partial_m \mathscr{S}_{nm}, \tag{1.1}$$

where δ_{nm} is the Kronecker delta function and the summation convention is implied unless otherwise stated. The continuity equation is $\partial_n \mathscr{U}_n = 0$ and the linearized boundary conditions to be applied to the mean flow are

$$\mathscr{U}_3 = U\,\partial_1\eta, \quad \mathscr{U}_1 = -U'\eta + \mathscr{U}_s \quad \text{at} \quad x_3 = 0, \tag{1.2}$$

where $\mathscr{U}_s$ is the fluctuating component of the mean tangential velocity of the surface. If $\mathscr{U}_s$ is of the same order of magnitude as the vertical velocity at the surface then it may be neglected in (1.2) so long as $U' \gg U$ at the surface. In most cases of laboratory or geophysical interest this condition is met and consequently $\mathscr{U}_s$ is neglected in the remainder of this paper.

In order to make use of these equations it is first necessary to relate the stresses $\mathscr{S}_{nm}$ to the other flow properties. This is the primary difficulty in predicting the turbulent flow over a wave owing to the fact that the dominant stresses are likely to be turbulent stresses for which no suitable constitutive equation has yet been found.

Miles (1957) assumed that the turbulent stresses were negligible throughout the flow and that viscous stresses were of importance only in the critical layer near points where $U = 0$ and very near the surface. This allows (1.1) to be reduced to the Rayleigh equation everywhere except in the critical layer. It is often said that the Miles mechanism is 'inviscid' but this statement is unfortunately misleading since it obscures the fact that work done on the surface, according to the theory, is determined entirely by the structure of the critical layer, which is dominated by viscous stresses. It will later be shown that, in fact, the work done on the boundary is entirely due to anisotropic stresses and that if the fluid were inviscid and laminar there would be no work done on the wave.

The confusion associated with the role of viscosity in the Miles model evidently stems from a misunderstanding concerning the process of matching the solutions of the Rayleigh equation across the critical layer, where that equation does not apply. For example, Lin (1955, ch. 8) expands the two fundamental solutions of the Rayleigh equation near the critical layer as

$$\psi_1(x_3), \quad \psi_2(x_3) = 1 + \ldots + (U_c''/U_c')\ln(x_3 - z_c)\,\psi_1,$$

where the subscript c refers to the critical height $x_3 = z_c$ at which $U = 0$. He then states that the "decision of the proper branch of the multiple-valued expressions is one of the main problems with solutions of this type". It is my opinion that this statement obscures the fact that the Rayleigh equation does not apply in the critical layer and therefore that the forms of ψ above and below the critical layer can be related only by matching each of them to the solution which is valid there. For $x_3 < z_c$

$$\psi_2 = 1 + \ldots + (U_c''/U_c')\,\psi_1[\ln|x_3 - z_c| + A]$$

is a solution of the Rayleigh equation. In general there is no reason why A, which is of critical importance in determining the work done on the wave, should be $(2n-1)\,i\pi$ as is implied by the choice of the 'proper branch' of the logarithm. When viscosity dominates in the critical layer, as was assumed by Miles, the appropriate constant is $A = -i\pi$. When nonlinear effects dominate over viscous ones a match across the critical layer is obtained by setting $A = 0$ (Davis 1969), and, if turbulent stresses were to dominate in the critical layer and be negligible outside, A may take on another value.

The importance of the critical layer rests in the fact that if the wave-induced stresses $\mathscr{S}_{nm}$ are nearly isotropic (as would generally be the case at large Reynolds number if the flow is laminar) it is only in the critical layer, where inertial effects are small, that the non-isotropic stresses can extract momentum from the mean flow. If, however, fluctuating turbulent stresses are accounted for, then non-isotropic stresses play an important role over much of the flow and the critical layer ceases to be of any particular significance.

Before turning to the problem of predicting the turbulent stress fluctuations $\mathscr{R}_{nm}$ it seems worthwhile to examine some more general aspects of flow over a wave in order to clarify the role played by non-isotropic stresses and to clarify certain points concerning momentum and energy fluxes to a wavy boundary. This is the subject of the following section.

2. Momentum and energy budget

Both Phillips (1966) and Miles (1967) have used the momentum flux through the wave surface to infer the growth rate of waves. These discussions have led to some controversy concerning both the magnitude of the overall average of the momentum flux and its importance in determining the rate of wave generation. In this section the flow of momentum and energy to the wave surface is discussed and the importance of anisotropic stresses in determining these quantities is demonstrated.

The mean flux of momentum through the surface is the scalar product of N_m, the unit normal to the surface, and the total momentum flow tensor $\hat{S}_{nm} - \hat{U}_n\hat{U}_m$. The mean flux of x_1 momentum is then

$$\hat{M}_1 = [S_{1m} + a\mathscr{S}_{13} - a(1+\delta_{1m})\,U\mathscr{U}_m]_\eta N_m,$$

where the subscript η denotes evaluation at the surface. Since the mass flux through the surface must vanish $\mathscr{U}_m N_m = 0$. Expanding about the mean surface

$x_3 = 0$, neglecting terms of $O(a^3)$ and taking an overall average gives the average momentum flow per unit projected horizontal area:

$$M_1 = S_{13}|_0 - a^2 \overline{\mathscr{U}_1 \mathscr{U}_3}]_0 + \frac{a^2}{2\pi} \int_0^{2\pi} [\eta\, \partial_3 \mathscr{S}_{13} - (\mathscr{S}_{11} - U\mathscr{U}_1)\, \partial_1 \eta]_0\, dx_1, \quad (2.1)$$

where the subscript 0 denotes evaluation at $x_3 = 0$. Integrating by parts and noting that $(\mathscr{S}_{11} - \mathscr{U}_1)\,\eta$ is periodic the integral in (2.1) becomes

$$\int_0^{2\pi} [-U_1 \mathscr{U}_1 + \partial_n \mathscr{S}_{1n}]_0\, \eta.$$

The terms in brackets is, according to (1.1), equal to $U'\mathscr{U}_3$ and from (1.2) it is seen that $\mathscr{U}_3$ and η are in quadrature and thus the integral vanishes. Therefore,

$$M_1 = [S_{13} - a^2 \overline{\mathscr{U}_1 \mathscr{U}_3}]_0 = [\mathscr{U}'/Re + R_{13} - a^2 \overline{\mathscr{U}_1 \mathscr{U}_3}]_0, \quad (2.2)$$

where $Re = U_0$ (wavenumber × kinematic viscosity) is the Reynolds number.

It will be noted that according to (2.2) the momentum flow to the wave, even in the presence of viscous and turbulent stresses and regardless of the boundary condition imposed on the turbulent stresses, is precisely $M_w = -a^3\,(\overline{\mathscr{U}_1 \mathscr{U}_3})_0$. This differs from the result given by Phillips (1966, equation (4.3.23)). This discrepancy arises because the momentum carried by stresses other than pressure were neglected by Phillips, a simplification which is inconsistent with the inclusion of deviatoric stresses in the final result. Equation (2.2) differs from the analogous result obtained by Miles (1967, equation (4.7*c*)) only in the inclusion of viscous effects.

It should be emphasized that (2.2) applies at the wave surface and not only outside the viscous sublayer. This fact leads to the rather surprising conclusion that the momentum flux to the wave is not dependent on the flow over the wave but depends only on the boundary condition imposed on this flow by the wave. Thus, once $\mathscr{U}_1$ at the mean wave surface is known, $-a^2(\overline{\mathscr{U}_1 \mathscr{U}_3})_0$ and hence the momentum flux to the wave are known irrespective of the nature of the flow. This result has the corollary that a purely irrotational periodic wave, for which $\mathscr{U}_1$ and $\mathscr{U}_3$ are in quadrature, cannot have its momentum increased by the action of the wind.

Evidently momentum flux is not a very useful measure of wave generation rate and it is therefore necessary to turn to the consideration of energy. The mean work done on the wave per unit area of the undisturbed surface is

$$E = a^2 \mathscr{S}_{nm} \mathscr{U}_n N_m,$$

and following standard techniques (cf. Batchelor 1967, § 3.4) it is possible to show that, to $O(a^2)$, the overall mean of this quantity is

$$E = a^2 \int_0^\infty \left[-\overline{\mathscr{U}_1 \mathscr{U}_3} \frac{dU}{dx_3} - \overline{\mathscr{S}_{nm}\, \partial_m \mathscr{U}_n} \right] dx_3. \quad (2.3)$$

The first term in this expression is the rate at which the energy of the mean flow is converted to kinetic energy associated with the wave-induced motion; the second term represents a conversion of wave-induced kinetic energy to heat and turbulent kinetic energy.

It is important to note that the relation $E = M_w c$ used by Miles (1967) to relate the energy flux to the overall mean of the wave-induced Reynolds stress is correct only when deviatoric stresses are confined to the critical layer and a thin boundary layer at the wave surface. In that case $\overline{\mathscr{U}_1\mathscr{U}_3}$ vanishes above the critical layer and is constant between the critical layer and the boundary layer. Neglecting the contribution to (2.3) from the regions where $\mathscr{S}_{nm}$ is important,

$$E = -a^2\int_0^\infty \overline{\mathscr{U}_1\mathscr{U}_3}\frac{dU}{dx_3} = \int_{U(0)}^0 M_w dU = M_w c. \tag{2.4}$$

Thus $E = M_w c$, where $M_w = -a^2\overline{\mathscr{U}_1\mathscr{U}_3}$ is evaluated not at the surface, but in the region between the critical and boundary layers.

As shown by Lighthill (1962), the overall mean stress $-\overline{\mathscr{U}_1\mathscr{U}_3}$ can be computed from

$$\overline{\mathscr{U}_1\mathscr{U}_3} = \int_\infty^{x_3} \overline{(\partial_3\mathscr{U}_1 - \partial_1\mathscr{U}_3)\mathscr{U}_3}\,dx_3, \tag{2.5}$$

where the integrand will be recognized as the correlation of the wave-induced vorticity and the vertical velocity. Taking the curl of (1.1), operating on the result by ∂_1, multiplying by $\mathscr{U}_3$ and taking an overall mean yields

$$-U\overline{(\partial_3\mathscr{U}_1 - \partial_1\mathscr{U}_3)\mathscr{U}_3} = \overline{[(\partial_3^2+1)\partial_1\mathscr{S}_{13} - \partial_3(\mathscr{S}_{11} - \mathscr{S}_{33})]\mathscr{U}_3}. \tag{2.6}$$

In arriving at this result the relation $\partial_1^2\mathscr{U}_n = -\mathscr{U}_n$ has been used. It is evident from (2.6) that if $\mathscr{S}_{nm}$ is isotropic, as would be the case if the flow were inviscid and laminar, then the correlation appearing in (2.5) must vanish everywhere except at the single point where $U = 0$. Thus, unless either the vorticity or $\mathscr{U}_3$ becomes infinite at that point, the stress $\overline{\mathscr{U}_1\mathscr{U}_3}$ must everywhere have the value found at infinity which presumably is zero. Thus for any physically realizeable inviscid laminar flow $\overline{\mathscr{U}_1\mathscr{U}_3} = 0$. Because $\partial_n\mathscr{U}_n = 0$ the energy flux (2.3) must also vanish.

The quasi-laminar model of Miles (1957), which is sometimes improperly referred to as inviscid, makes use of the asymptotic solution of the Orr–Sommerfeld equation in the limit of vanishing viscosity to evaluate $\overline{\mathscr{U}_1\mathscr{U}_3}$. The nearly discontinuous variation of this quantity found in the critical layer must, according to (2.6), result from the viscous stresses which dominate in this region.

From the foregoing discussion it is evident that the work done on a wave cannot be predicted unless the anisotropic stresses associated with the flow are accurately known. Since these stresses are likely to be primarily turbulent Reynolds stresses this presents an extreme difficulty owing to the absence of any established method of predicting turbulent stresses. The approach adopted here is to examine two different models of turbulent stress variations and to apply them to flow over a wavy boundary. The two models are discussed in §§ 3 and 4 and the results of the models are compared with experiment in § 5.

3. Stress conservation equations

Perhaps the most straightforward method of dealing with turbulent stress fluctuations is to attempt a phenomenological closure of the Reynolds stress conservation equations. Using standard techniques (Hinze 1959) it is possible to derive and linearize the equations for the Reynolds stresses into the form

$$U\,\partial_1 \mathscr{R}_{nm} + R_{nk}\,\partial_k \mathscr{U}_m + R_{km}\,\partial_k \mathscr{U}_n + U'(\delta_{1m}\mathscr{R}_{n3} + \delta_{1n}\mathscr{R}_{3m}) = \mathscr{T}_{nm} + (1/Re)\,\partial_k^2 \mathscr{R}_{nm}, \tag{3.1}$$

where, in keeping with the discussion of § 1, $\partial_3 R_{nm}$ has been taken as zero and Re is the Reynolds number. The term $\mathscr{T}_{nm}$ is the fluctuating part of $\hat{T}_{nm}$, the mean of

$$\partial_k u_n u_m u_k + u_n \partial_m p + u_m \partial_n p + (2/Re)\,\partial_k u_n \,.\, \partial_k u_m.$$

In order to make use of (3.1) to predict turbulent stress fluctuations one must find a way of relating $\mathscr{T}_{nm}$ to the quantities $\mathscr{R}_{nm}$ and $\mathscr{U}_n$. Bradshaw *et al.* (1967) have proposed such a closure hypothesis, which has resulted in remarkably successful predictions of boundary-layer development under various impressed pressure gradients. The basic idea involved in this method is that the value of $\hat{T}_{nm}$ is more closely related to the turbulence itself than to the mean flow and may therefore be predicted from the local value of $\hat{R}_{nm}$. The success of the Bradshaw, Ferriss & Atwell (BFA) model suggests that it may be useful in predicting flow over a wave, but before this can be done certain modifications of the model must be made. First, all dependence on the boundary-layer thickness must be removed since for geophysical flows there is no boundary-layer thickness and the only characteristic scale is the height above the boundary. Further, the BFA model considers only the shear stress $\hat{R}_{13}$ (the only stress of importance in boundary-layer calculations) whereas for flow over a wave the normal stress components must be considered; the model must therefore be expanded to include prediction of these stresses. The most serious difficulty with the BFA model is the assumption that the shear stress and the average normal stress (or the turbulent kinetic energy) vary proportionally. This assumption is at variance with Kendall's (1970) observation that the shear stress and the turbulent 'intensity' appear to exhibit different variations with wave phase. The approach adopted here is to test two generalizations of the BFA model, one in which all stresses are assumed to vary proportionally and one which considers that only the normal stress variations are proportional. In either case a relation between $\hat{T}_{nm}$ and $\hat{R}_{nm}$ is assumed on the basis of dimensional arguments and the constants in the relation are adjusted so as to be in agreement with the known behaviour of a constant stress boundary layer. The models, which then contain no adjustable constants, are used to compute flow over a wave and the results compared with experiment in § 5.

For both models it is assumed that the normal stress variations are proportional to variations of the average normal stress $\hat{Q} = \hat{R}_{nn} = Q + a\mathscr{Q}$ according to

$$\mathscr{R}_{nm} = \mathscr{Q} R_{nm}/Q \quad \text{for} \quad n = m.$$

Summing (3.1) over $n = m$ leads to

$$U\partial_1 \mathscr{Q} + 2R_{nk}\partial_k \mathscr{U}_n + 2U'\mathscr{R}_{13} = \mathscr{T}_{nn}. \tag{3.2a}$$

The conservation equation for shear stress is

$$U\,\partial_1 \mathscr{R}_{13} + R_{11}\partial_1 \mathscr{U}_3 + R_{33}\partial_3 \mathscr{U}_1 + U'(R_{33}/Q)\,\mathscr{Q} = \mathscr{T}_{13}. \tag{3.2b}$$

In both of these equations the molecular diffusion of Reynolds stress ($Re^{-1}\partial_k^2 \mathscr{R}_{nm}$) has been neglected, thus restricting the model to high Reynolds number flows.

The quantity $\hat{T}_{nm}$ has the dimensions of (velocity)^3 × (length)$^{-1}$ and on dimensional grounds it has been assumed that

$$\hat{T}_{nn} = \frac{2}{3}\frac{\sigma_n}{L}|\hat{R}_{nn}|^{\frac{3}{2}}, \quad \hat{T}_{13} = \frac{2}{3}\frac{\sigma_s}{L}\hat{R}_{13}^{\frac{3}{2}},$$

where L is the height above the boundary, that is, $L = x_3 - a\eta$. Linearizing the expressions for $\hat{T}_{nm}$ gives

$$\mathscr{T}_{nn} = \frac{\sigma_n}{x_3}|\hat{Q}|^{\frac{1}{2}}\left[-\mathscr{Q} + \tfrac{2}{3}Q\frac{1}{x_3}\eta\right], \tag{3.3a}$$

$$\mathscr{T}_{13} = \frac{\sigma_s}{x_3}R_{13}^{\frac{1}{2}}\left[\mathscr{R}_{13} - \tfrac{2}{3}R_{13}\frac{1}{x_3}\eta\right]. \tag{3.3b}$$

The constants σ_n and σ_s can be found by requiring (3.3) to predict $\mathscr{T}_{nm}$ when the perturbation stress $\mathscr{R}_{nm}$ is a constant, that is, when there is a small change of the stress in a boundary layer with vertically uniform shear stress. If the reference velocity U_0 is taken as the shear velocity over von Kármán's constant, the law of the wall, which pertains to the particular stress perturbation considered here, yields

$$\mathscr{U}_1 = 2(\mathscr{R}_{13}/R_{13})\ln x_3/z_0.$$

Substitution of the form of $\mathscr{U}_1$ and equations (3.3) into (3.2) then gives

$$-|Q|^{\frac{1}{2}}\sigma_n = 3R_{13}/Q, \quad R_{13}^{\frac{1}{2}}\sigma_s = \tfrac{3}{2}R_{33}/R_{13}.$$

One of the models tested here is essentially a direct adaptation of the BFA prediction scheme in which it is assumed that all turbulent stresses vary proportionally. Thus for model A it is assumed that

$$\mathscr{R}_{nm} = \mathscr{Q}R_{nm}/Q \tag{3.4}$$

for both shear and normal stresses. This assumption is not consistent with the simultaneous satisfaction of both of (3.2) and the two relations (3.3). In keeping with the BFA scheme it has been assumed that (3.2*a*) and (3.3*a*) are applicable but (3.2*b*) has been dispensed with.

The second model tested here (model B) results from assuming that the normal stresses obey (3.4) but that the shear stress does not. In this case all the equations (3.2) and (3.3) are used to calculate both $\mathscr{R}_{13}$ and $\mathscr{Q}$.

The Reynolds stress equations together with the momentum equations (1.1) can be reduced to a set of ordinary differential equations by letting

$$(\mathscr{U}_1, \mathscr{U}_3, \mathscr{P}, \mathscr{R}_{nm}) = \mathrm{Re}\,(\psi', -i\psi, \pi, r_{nm})\,e^{ix_1}.$$

The momentum equations then become

$$Ui\psi' - iU'\psi + i\pi = r_{13}' + ir_{11}, \tag{3.5a}$$

$$U\psi + \pi' = r_{33}' + ir_{13}, \tag{3.5b}$$

when it is assumed that $\mathscr{S}_{nm} = -P\delta_{nm} + \mathscr{R}_{nm}$. The stress conservation equations appropriate to model A become

$$(Ui - U'R_{13}/Q)\,q = -2R_{13}U'' - 2[R_{13}(\psi'' + \psi) + (R_{11} - R_{33})\,i\psi'], \tag{3.6a}$$

$$r_{nm} = R_{nm}q/Q \tag{3.6b}$$

and those for model B are

$$(Ui - 3U'(R_{13}/Q))\,q - 2U'r_{13} = -2R_{13}U'' - 2[R_{13}(\psi'' + \psi) + (R_{11} - R_{33})\,i\psi'], \tag{3.7a}$$

$$(Ui - \tfrac{3}{2}(R_{33}/R_{13})\,U')\,r_{13} + (R_{33}/Q)\,U'q = -R_{33}U'' - R_{33}\psi'' - R_{11}\psi. \tag{3.7b}$$

Before proceeding to the solution of these equations it appears worthwhile to point out certain features of the stresses predicted by models of this general type. If the $\mathscr{T}_{nm}$ functions in (3.1) are chosen to be proportional to $\mathscr{R}_{nm}$ and it is required that the forms be consistent with the law of the wall it follows that as x_3 approaches infinity $\mathscr{T}_{nm}$ varies as $O(\mathscr{R}_{nm})/x_3$ and therefore (3.1) simplifies to

$$U\partial_1\mathscr{R}_{nm} = -R_{nk}\partial_k\mathscr{U}_m - R_{mk}\partial_k\mathscr{U}_n, \tag{3.8}$$

if molecular diffusion of $\mathscr{R}_{nm}$ is neglected. This behaviour has two important consequences. First, the assumption that $\mathscr{R}_{nm} = R_{nm}\mathscr{Q}/Q$ is not generally valid; but more importantly, because of the pseudo-elastic behaviour exhibited by (3.8), the form of ψ as $x_3 \to \infty$ is entirely different from the corresponding behaviour for a viscous fluid. When the stresses are given by a viscous law (3.5) admits two solutions which vanish as $x_3 \to \infty$; when the stresses are given by (3.8) there is only one such solution and consequently only one of the boundary conditions at the surface can be applied.

In order to examine the flow far from the boundary it is convenient to eliminate π from (3.5) and to substitute (3.8) for the stress functions r_{nm}. When terms of $O(1/x_3)$ are neglected this leads to

$$\begin{aligned} iU(\psi'' - \psi) &= r_{13} + i(r_{11}' - r_{33}') \\ &= \frac{i}{U}\left[R_{33}\frac{d^2}{dx_3^2} + 2iR_{13}\frac{d}{dx_3} - R_{11}\right](\psi'' - \psi). \end{aligned}$$

Since $U' \ll 1$ the solutions of this equation can be approximated by taking U constant and letting $\psi = \exp(\alpha x_3)$. The four possible values of α are

$$\alpha = \pm 1, \quad \alpha = -i\frac{R_{13}}{R_{33}} \pm \frac{1}{R_{33}}[R_{33}U^2 + R_{33}R_{11} - R_{13}^2]^{\frac{1}{2}}.$$

When $U^2 \gg |R_{nm}|$ two of these roots are pure imaginary and apparently are associated with a shear wave supported by the pseudo-elastic behaviour of the stress relation (3.8). Models A and B give similar results. In each case when

$U^2 \gg |R_{nm}|$ there are two imaginary roots and one with a negative real part; the roots are different for the two models and in each case they differ from the results obtained using (3.8), but the one root of interest in each case approaches -1 as $R_{nm}/U^2 \to 0$.

The fact that only one boundary condition can be applied at the surface indicates that equations (3.5) and the stress predictions based on our approximation to (3.1) do not adequately model the flow. The inclusion of direct viscous stresses in (3.1) will not alter this result when $Re \gg 1$. However, if the molecular diffusion of Reynolds stress is retained in equations (3.2) the differential system will be raised to sixth order. The additional solutions will be associated with characteristic α values of $O(Re^{\frac{1}{2}})$. One of these additional solutions could be used to construct a boundary-layer solution which would satisfy the tangential boundary condition at the surface in the same way as the rapidly decaying 'viscous' solution of the Orr–Sommerfeld equation is used to satisfy that condition in laminar flow problems. We are led, then, to the conclusion that although the turbulence model advanced here includes anisotropic stresses and represents a fourth-order system it plays the same role in predicting turbulent flow as the inviscid Orr–Sommerfeld equation plays in laminar flow problems. However, in contrast to the inviscid Orr–Sommerfeld equation, the equations associated with the turbulence model are regular wherever U' is bounded.

One additional point of interest is the behaviour of the solutions of (3.1)–(3.3) when horizontal derivatives are small compared with vertical derivatives, the situation that is expected to arise near the wave surface. It is not difficult to see that in this case

$$U + a\mathscr{U}_1 = \left(1 + \frac{a}{2}\frac{\mathscr{R}_{13}}{R_{13}}\right)\ln\left(\frac{x_3 - a\eta}{z_0}\right),$$

$$\mathscr{R}_{13} = \frac{2R_{13}}{U'}\partial_3\mathscr{U}_1, \quad \mathscr{Q} = \frac{2Q}{U'}\partial_3\mathscr{U}_1$$

describe the flow to $O(a)$. The behaviour of $\mathscr{R}_{13}$ is identical with the eddy viscosity model proposed by Hussain & Reynolds (1970) but the behaviour of $\mathscr{Q}$ is not; in fact this type of stress variation cannot be obtained from an eddy viscosity constitutive equation if the viscosity is taken as a scalar rather than the more general second-order tensor appropriate to an anisotropic medium. The velocity $U + a\mathscr{U}_1$ will be recognized as the linearized law of the wall for a bounding surface at $x_3 = a\eta$.

Finally, the method of applying the surface boundary condition $\psi_0 = -U(0)$ deserves some comment. Up to now it has been assumed that the primary velocity profile is logarithmic, but this cannot be correct down to $x_3 = 0$, where it is known that $U = -c$ (c being the wave speed). Without precise information about U very near the surface and additional assumptions about the fluctuating turbulent stress in that region it is not possible to integrate the equations to $x_3 = 0$, where the surface condition is to be applied. However from the discussion above it is seen that near the surface (but in the region where U is still logarithmic)

$$\mathscr{U}_1 = -U'\eta + \tfrac{1}{2}(\mathscr{R}_{13}/R_{13})\,U.$$

If it is assumed that this behaviour continues down to the surface it is possible to develop a boundary condition on $\mathscr{U}_3(h)$, where h is some small height where the profile is logarithmic. To do this the continuity equation is integrated from $x_3 = 0$ according to

$$\begin{aligned}\mathscr{U}_3(h) &= \mathscr{U}_3(0) - \int_0^h \partial_1 \mathscr{U}_1 \, dx_3 \\ &= U(h)\, \partial_1 \eta - \frac{1}{2}\int_0^h \frac{U}{R_{13}} \partial_1 \mathscr{R}_{13} \, dx_3.\end{aligned}$$

If $h \ll 1$ then the second integral contributes little (this assumption has been verified from the results) and it is found that the appropriate surface condition on ψ is $\psi(h) = -U(h)$. The results in § 5 are based on the assumption that h is the position where $U(h) - U(0) = 1$ but these results can hardly be distinguished from those obtained using larger values of the velocity difference across the sublayer.

4. Viscoelastic turbulence

The concept of an eddy viscosity is neither new nor is it pleasing from an aesthetic point of view. Examination of the stress conservation equations (3.1) suggests it is unlikely that the terms $\mathscr{T}_{nm}$ will adjust in just such a way that the stress is proportional to the local rate of strain. It seems that, at the very least, any attempt to propose a phenomenological relation between stress and mean flow should allow for elastic behaviour of the turbulence. Nevertheless, Hussain & Reynolds (1970) have had considerable success in describing the dynamics of perturbations to a turbulent flow using an eddy viscosity model. It therefore seems worthwhile to examine the consequences of applying an eddy viscosity, or perhaps a combination of an eddy viscosity and elasticity, to the flow over a wave.

The models described in this section are based on the assumption that the turbulent stress of a fluid element is determined by the rate of strain it has recently experienced. Therefore it is proposed that

$$\hat{R}_{nm} - \tfrac{1}{3}\delta_{nm}\hat{R}_{kk} = \int_{-\infty}^{t} H(t,\tau)\,[\partial_n \hat{U}_m + \partial_m \hat{U}_n]\, d\tau, \tag{4.1}$$

where the rate of strain is to be evaluated along the path $\dot{x} = \hat{U}$. It can be seen from (1.1) that the normal stress acting on the surface $\mathscr{R}_{33} - \mathscr{P}$ is not affected by the quantity $\mathscr{R}_{kk}$ and therefore the average normal turbulent stress need not be determined. If the 'memory' function $H(t,\tau)$ vanishes rapidly as $t - \tau \to \infty$ then the stresses predicted by (4.1) will be primarily determined by the local rate of strain and the fluid will behave in a viscous manner. However if the fluid's memory is long the constitutive relation (4.1) will be that of a viscoelastic fluid.

One constraint on possible memory functions results from requiring (4.1) to agree with the 'law of the wall' relation for small changes in $\hat{R}_{13}$. This leads to

$$\int_{-\infty}^{t} H(t,\tau)\, d\tau = 2R_{13}/U'. \tag{4.2}$$

This relation alone is insufficient to adequately specify H and so it has been assumed that H is of the simple form

$$H(t,\tau) = 2R_{13}\omega\, e^{U'\omega(\tau-t)},$$

which satisfies (4.2) and has only the one adjustable parameter, ω. This simple form allows (4.1) to be integrated directly. The fluid element which passes through the point (x_1, x_3, t) follows the path

$$X_1(\tau) = x_1 + [\tau - t]\, U + O(a),$$
$$X_3(\tau) = x_3 - a(\psi/U)\,[e^{iU(\tau-t)} - 1]\, e^{ix},$$

where $\psi(x_3)$ is as defined just above (3.5). Substituting this into (4.1) gives

$$\mathscr{R}_{13} = \frac{2R_{13}}{U' + iU/\omega}\left[\psi'' + \psi - \frac{iU'}{\omega}\psi\right] e^{ix_1}, \tag{4.3}$$
$$\mathscr{R}_{11} - \mathscr{R}_{33} = \frac{2R_{13}}{U' + iU/\omega}[4i\psi']\, e^{ix_1}.$$

The results labelled model C in § 5 correspond to the choice $1/\omega = 0$. In this case the shear stress behaviour is identical to that of a viscous fluid with the viscosity $2R_{13}/U'$ and is therefore identical to the form for $\mathscr{R}_{13}$ proposed by Hussain & Reynolds.

The final model tested in this paper is obtained by requiring that in the limit $U' \to 0$ the constitutive relation should approximate the behaviour of (3.8). This behaviour cannot be modelled exactly by (4.1) but an order of magnitude agreement can be achieved if one is willing to approximate (3.8) by

$$U\partial_1\mathscr{R}_{13} = -\tfrac{1}{2}(R_{11} + R_{33})\,(\partial_3\mathscr{U}_1 + \partial_1\mathscr{U}_3).$$

This pseudo-elastic behaviour is obtained from (4.3) when $\omega = -4R_{13}/(R_{11} + R_{33})$; this value of ω was used for the results which in the next section are referred to as model D.

An analysis similar to that outlined in § 3 can be used to show that when the fluctuating stresses are given by (4.3) the momentum equations (1.1) admit two solutions which vanish as $x_3 \to \infty$. Therefore an eddy viscoelasticity model can, in contrast to the models discussed in the previous section, satisfy both the surface boundary conditions (1.2). Unfortunately this presents a serious difficulty since the value of the horizontal velocity component $\mathscr{U}_1$ at the surface is determined by U', the gradient of the primary flow at the surface. The logarithmic velocity profile predicted by the law of the wall is not correct at the surface and there are no measurements of U' very near a wavy surface. The lack of a precise primary velocity profile near the surface is not a serious problem in dealing with models A and B since only the fluctuating vertical velocity $\mathscr{U}_3$ is required and this varies much more slowly than $\mathscr{U}_1$, however for the eddy viscoelasticity models some way of dealing with this problem must be found.

As will be seen from the results to be presented in § 5 the details of the wave-induced flow predicted by models C and D are strongly dependent on the details of the mean velocity profile very near the surface. Davis (1970) found that the quasi-laminar model is also sensitive to the form of U very near the surface.

Because we have no data on the mean velocity near the surface, models C and D can only be properly tested by applying them to flow over waves under various velocity profiles all of which become logarithmic far above the surface. The particular class of profiles investigated here are defined by

$$U = \ln(x_3/z_0) - c = \ln(x_3/z_c) \quad \text{for} \quad x_3 > z_1,$$
$$U = (x_3/z_1)\ln(z_1/z_0) - c \quad \text{for} \quad 0 < x_3 < z_1.$$

This profile is continuous at the transition point z_1. Across the sublayer the velocity increases linearly by the amount $\Delta U = \ln(z_1/z_0)$ and it is this quantity which is used in § 5 to characterize the individual members of the velocity profile family.

5. Results

The results presented in this section were obtained through numerical integration of the ordinary differential equations relevant to each different turbulence model tested (equations (3.5) and (3.6*a*) for model A, (3.7) for model B and (4.3) for models C and D). The equations were integrated from $x_3 = 10$ towards $x_3 = 0$ using a fourth-order Runge–Kutta algorithm and a step size of $-0{\cdot}02$ for $x_3 > 1{\cdot}0$ and $-0{\cdot}02x_3$ for $x_3 < 1$. The choice of step size and starting ordinate were determined through experimentation using the acceptance criteria that a 50 % increase in the starting ordinate or a 50 % reduction in step size should produce a change of less than 0·05 %. An inability to achieve these criteria with the equations for models A and B led, in fact, to the discovery that the fourth-order differential systems appropriate to these models admit only one solution which vanishes as $x_3 \to \infty$. Two independent solutions for the eddy viscoelasticity models were obtained without the use of the 'filtering' techniques required to solve the Orr–Sommerfeld for high Reynolds numbers.

The principal results are presented in the form of the coefficients α and β used by Miles (1957) to describe the normal stress on the boundary. These are defined by

$$\hat{P} - \hat{R}_{33} = a(\alpha + i\beta)\, e^{ix_1} + P - R_{33},$$

where it must be remembered that stresses are scaled by density $\times\, U_0^2$ and the parameter a is the dimensionless wave amplitude, which is equal to the maximum wave slope. It is important to note that it is the total normal stress which is nearly constant across the sublayer near the surface; if the surface is smooth $\hat{R}_{33}$ must vanish on the surface and although both $\hat{P}$ and $\hat{R}_{33}$ may vary rapidly the total normal stress will not. In any event, it is the total stress which does work on the wave.

For comparison the experimental results of Dobson (1969) and Kendall (1970) are summarized in figure 1. Dobson kindly provided values of β computed from his data and these are plotted as a numeral (the value of β) next to a symbol which is used to code the run from which the value is taken and to denote the values of z_c, the dimensionless critical height, and c, the dimensionless wave speed. Kendall's results are much less variable than Dobson's and a summary of them is contained in the six plotted points; almost all his experimental values of β fall

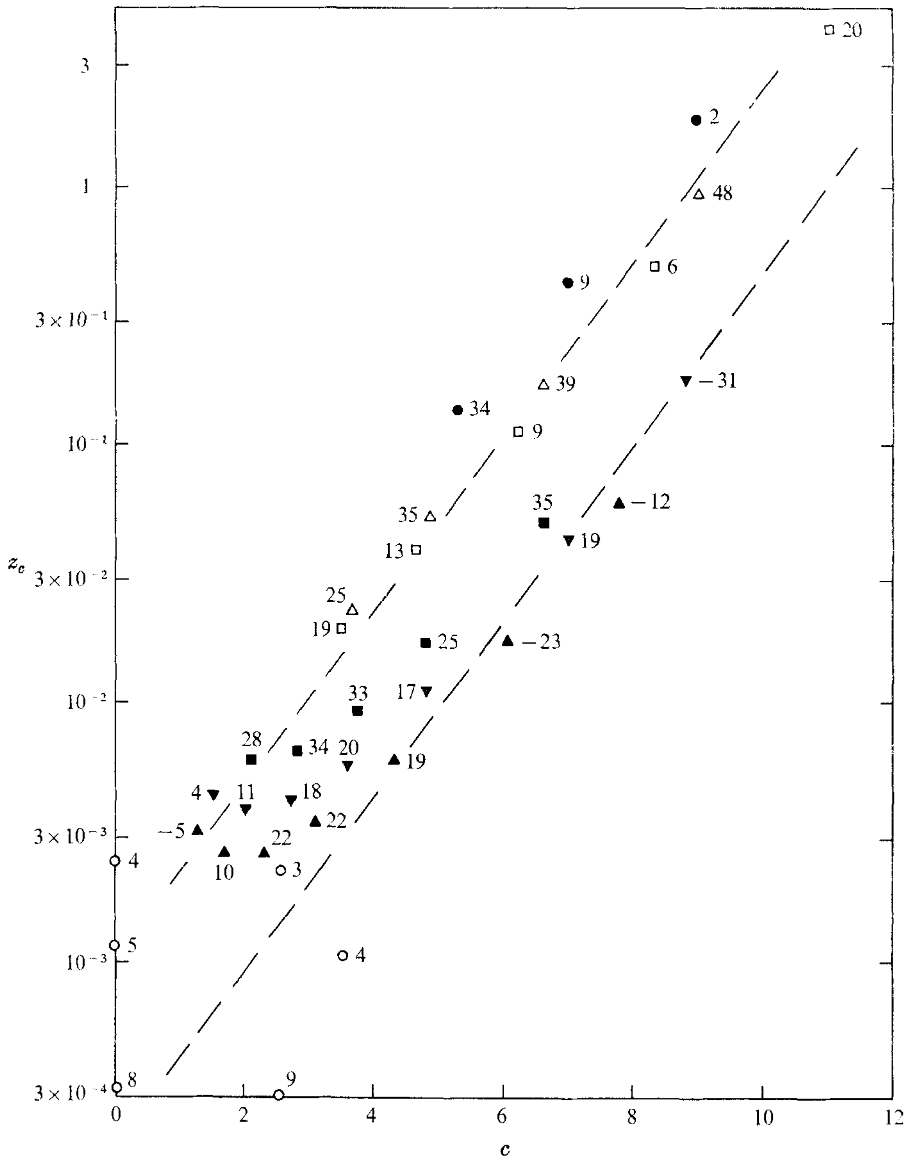

FIGURE 1. Experimental values of β (the numerals beside the symbols) shown on a plot of z_c *vs*. c. Dobson's results: ●, run 1; △, run 2*a*; ▽, run 2*b*; □, run 3; ▲, run 4*a*, ▼, run 4*b*; ■, run 6. ○, typical values obtained by Kendall.

between 3 and 10 and have the general trend indicated by the points plotted. In order to facilitate comparison of measured and predicted values of β the two parallel dashed lines in figure 1 will be plotted along with the predicted values.

It should be pointed out that figure 1 must be interpreted in the light of the fact that the co-ordinates z_c and c are not measured directly and that their computation involves use of the roughness height of the mean logarithmic profile, a quantity which is rarely known with high accuracy. Another important feature bearing on the comparison of β values obtained under different conditions is the fact that according to the eddy viscoelasticity models of § 4 the wave-induced flow is sensitive to the details of the mean flow very near the surface. If this type of model is correct then one would not expect close agreement of data obtained in the field and the laboratory since in the laboratory the region influenced by viscosity is apt to be significantly more extensive than in the field and this will be likely to be manifest in differences between the mean velocity profiles near the surface. It will be noticed that Dobson's values of β are fairly consistent for $c < 5$ but for larger values of c they appear to vary rapidly without any obvious relation to the parameters c and z_c. It is not known to what extent this is due to experimental difficulties but it will be seen that eratic variation of β is not inconsistent with the predictions of the eddy viscoelasticity models C and D.

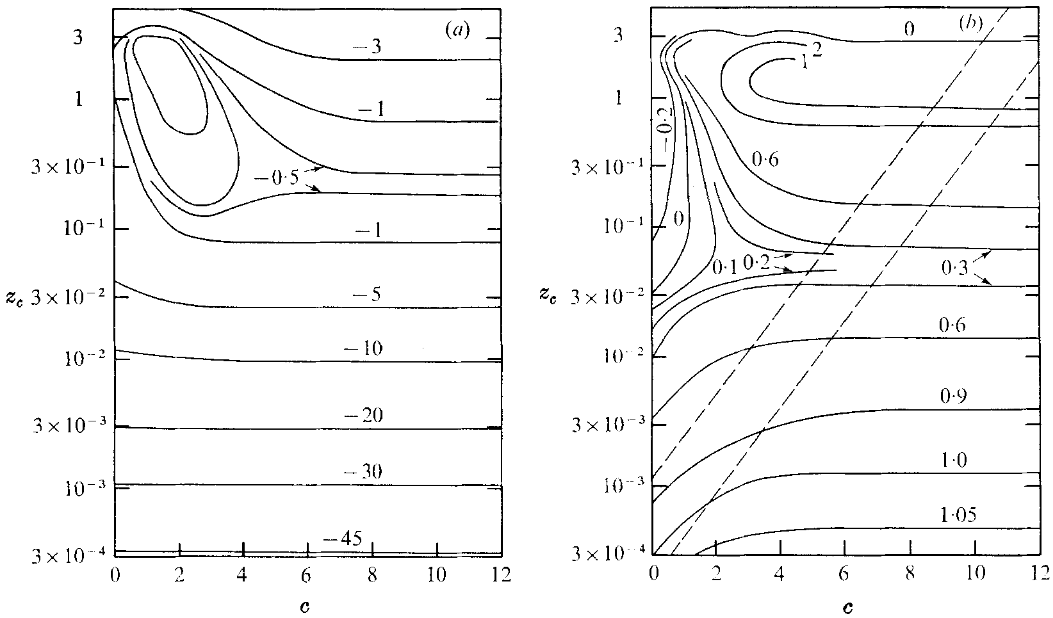

FIGURE 2. Contour plots of (a) α and (b) β predicted by model A.

Figures 2–4 are contour plots of the normal stress coefficients α and β predicted by the various turbulence models. Before discussing these results the method of preparing the contour plots deserves some comment. Since it is relatively economical to obtain integrations for various values of c and ΔU (the value of the mean flow difference across the surface sub-layer) while holding z_c constant, the contour plots have been constructed from 'sections' along which z_c is constant. The variation of α and β between these sections (which are at $z_c = 0{\cdot}001, 0{\cdot}003, \ldots, 1, 3$) has been inferred and must therefore be interpreted with some caution.

The results obtained from model A (the adaptation of the Bradshaw, Ferriss & Atwell model) are contained in the contour maps of α and β depicted in figure 2. For these calculations $R_{11} = -0{\cdot}80$, $R_{22} = -0{\cdot}35$, $R_{33} = -0{\cdot}10$ and $R_{13} = 0{\cdot}16$. These values seem to be a reasonable compromise between the data presented by Lumley & Panofsky (1964) and Volkov (1969). The predicted values of β are of

$O(1)$ rather than $O(10)$, which appears to be representative of the experimental values. A local minimum value of β occurs around $z_c = 0{\cdot}05$ and β is negative for $z_c > 3$. Before it was realized that only one boundary condition should be applied to solutions for model A some results were obtained using both normal and tangential velocity boundary conditions at the surface. The predicted values of β were large (approximately of the size obtained for models C and D) apparently owing to the fact that U', which enters into the tangential velocity boundary condition, is generally large.

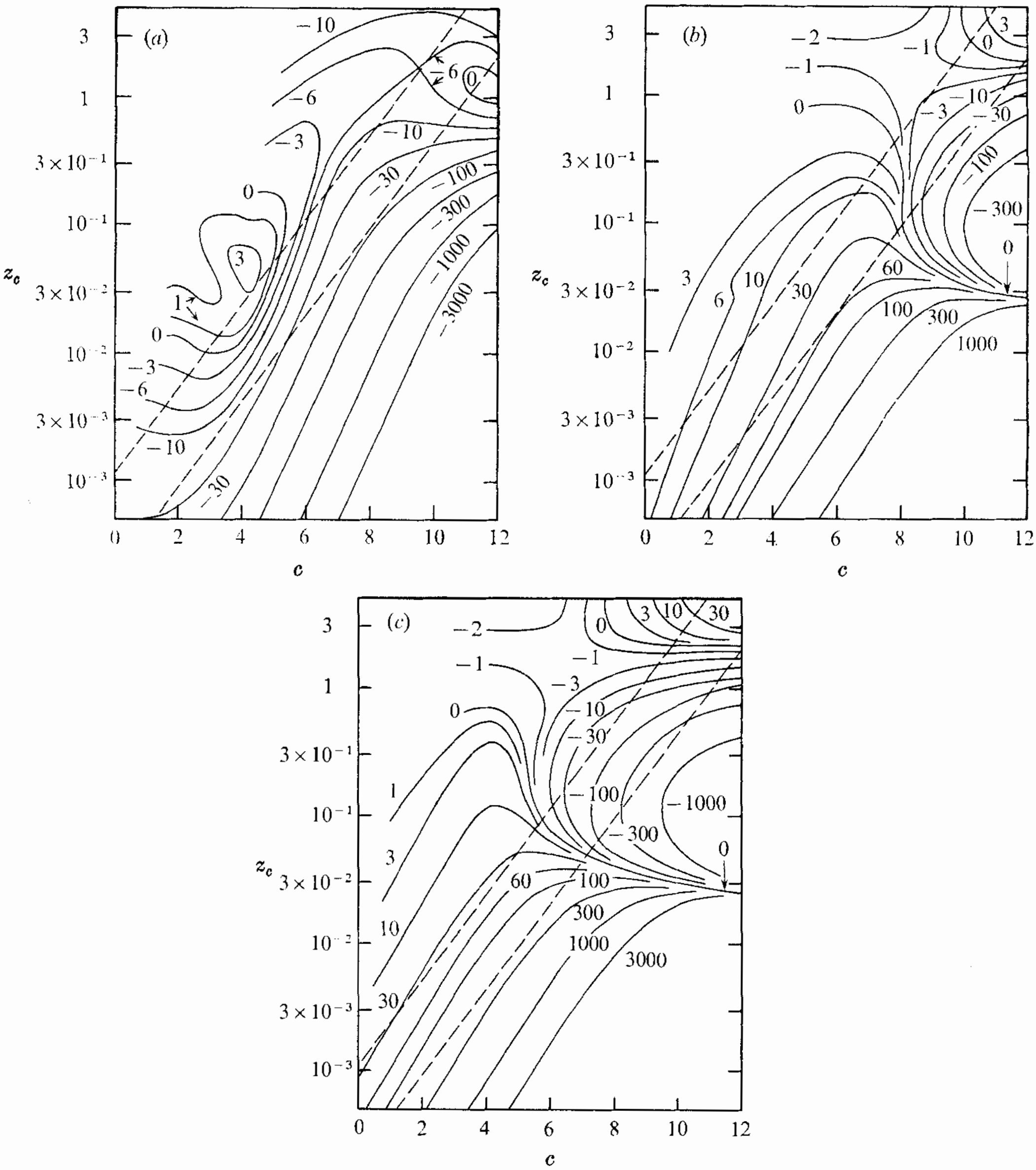

Figure 3. Contour plots of (*a*) α, (*b*) β and (*c*) β predicted by model C for $\Delta U = 5$, 5 and $2{\cdot}5$ for (*a*), (*b*) and (*c*) respectively.

Of the four models tested here the least satisfactory was B (the modification of model A in which $\mathscr{R}_{13}$ and $\mathscr{R}_{nn}$ vary independently) and no results for this model have been presented. Most of Kendall's and Dobson's measurements show $\beta > 0$ and $\alpha < 0$; this situation was almost never found for model B and no amount of imagination could result in a favourable comparison of theory and experiment.

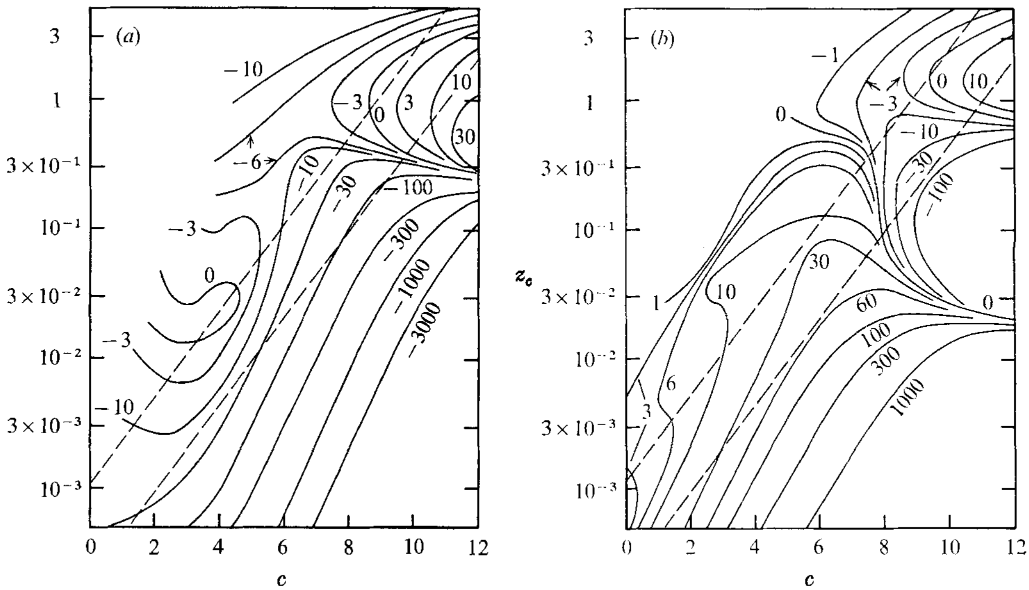

FIGURE 4. Contour plots of (*a*) α and (*b*) β predicted by model D for $\Delta U = 5$.

The values of α and β predicted by the eddy viscoelasticity models C and D are presented in figures 3 and 4. These two models differ primarily in the relationship of stress and rate of strain far from the wave surface. Near the surface, $U' \gg U/\omega$ and the constitutive relation (4.3) is approximately the same for the two models; it is only far from the surface, where $U \simeq U'$, that the value of ω is important. Intercomparison of figures 3(*a*) and 4(*a*) (α for models C and D with $\Delta U = 5$), and figures 3(*b*) and 4(*b*) (β for the same conditions) shows that the two models do not differ significantly. This suggests that the surface normal stress is determined primarily by the nature of the flow near the boundary, where the two models predict similar relations between the fluctuating stresses and the mean flow.

Comparison of figures 3(*b*) and (*c*) (β for model C with $\Delta U = 5$ and 2·5, respectively) demonstrates the importance of the mean velocity profile near the surface. It will be noticed that the principal difference between these two figures can be ascribed to an apparent displacement of the $c = 0$ axis: if the abscissa of figure 3(*c*) ($\Delta U = 2{\cdot}5$) were replaced by the value of $c+2$ the figure would be very similar to figure 3(*b*). This relationship was found for α and β as predicted by both models and appears to follow from the fact that flow is primarily determined by the tangential surface boundary condition $\mathscr{U}_1(0) = -U'\eta$. The surface shear is $U' = (\Delta U/z_c) \exp(c - \Delta U)$, which remains constant when ΔU is changed from 2·5 to 5 and c is reduced by 1·8.

It will be noticed that according to both models C and D there is a region of negative β for large c when $z_c \simeq 0{\cdot}01$. Results obtained using many values of the memory time constant parameter ω indicate that this is a universal feature of eddy viscoelasticity models and it is interesting to note that this region of wave damping occurs at roughly the value of z_c at which model A predicts a minimum in β. It should also be noted that Dobson obtained some large negative values of β in this region.

In addition to the above comparison of predicted and measured surface pressures it would be desirable to compare other details of the flow. An attempt has been made to compare the wave-induced mean flow components $\mathscr{U}_1$ and $\mathscr{U}_3$ predicted by models C and D with the laboratory measurements of Stewart (1970). Unfortunately, these features of the flow are more sensitive to the nature of the mean velocity profile near the surface than is the surface pressure. This makes it difficult to draw any definite conclusions from comparisons with experiment since the mean profile is unknown.

Despite the uncertainty associated with the sublayer region of the mean velocity profile certain general comments can be made about the predicted and measured values of $\mathscr{U}_1$ and $\mathscr{U}_3$. The choice $\Delta U = 5$ appears to give the best values of β and so most calculations were made for that choice. In general the predicted magnitudes of $\mathscr{U}_1$ and $\mathscr{U}_3$ exceed the measured values. The magnitudes predicted by D exceed those of C for $x_3 > z_c$ and are generally smaller for $x_3 < z_c$. The agreement for $\mathscr{U}_1$ is about the same that as found by Stewart (1970) and Davis (1970) for the quasi-laminar model. Both models C and D overestimate $\mathscr{U}_3$ somewhat more than the quasi-laminar model. As was found for the quasi-laminar model the phases of $\mathscr{U}_1$ and $\mathscr{U}_3$ are extremely sensitive to the sublayer profile and no real comparison is possible other than to say that model C compares at least as well as the quasi-laminar model. Rapid oscillation of the phases of $\mathscr{U}_1$ and $\mathscr{U}_3$ predicted by D are found for $x_3 > 3$ but Stewart's measurements do not include this region.

6. Conclusions

The results presented above suggest some conclusions not only about the success of predicting flow over waves using the particular turbulence models tested but also about what additional studies are required to allow further progress.

First, it may be said that the surface normal stress predicted by the eddy viscoelasticity models is in reasonable agreement with experimental values. As was shown in § 2 the work done on the surface by the wave-induced mean stress is in fact the result of an interaction of the mean flow and anisotropic stresses. In Miles's (1957) quasi-laminar model these stresses are assumed to be direct viscous stresses; this assumption results in predicted β coefficients which are much smaller than those measured. At the very least, the results obtained here show that fluctuating turbulent stresses can explain the large measured values of β. Further, they appear to indicate that the tangential velocity boundary condition on the wavy surface plays an important role in the flow and models, such as those of § 3, which do not account for this condition will be unsatisfactory in determining the work done on the wave. Because the important tangential velocity boundary condition involves the primary flow shear U' at the surface the mean velocity profile over waves must be determined down to the region under wave crests. This poses a very challenging experimental problem but one which is essential to further progress.

It is also evident that the behaviour of such parameters as α and β cannot be characterized by a single parameter such as wave velocity over wind speed or critical layer height over wavelength; the surface stress depends on both these parameters and the dependence is not necessarily a slow one. This means that fairly precise data concerning the mean profile are required. Further, measurements which use frequency filtering to determine wave-induced quantities must be of fairly high frequency resolution since c and z_c are strong functions of the wave frequency.

Finally, it must be emphasized that measurements of the wave-induced mean velocities $\mathscr{U}_1$ and $\mathscr{U}_3$ are not very useful either for testing the predictions of a theory or for determining the work done on the wave. If, as now seems certain, anisotropic stresses play an important role in the flow the wave-induced mean Reynolds stress $-\overline{\mathscr{U}_1\mathscr{U}_3}$ is of very little relevance to the work done on the wave. As was pointed out in § 5, the mean velocity components appear to be very much more sensitive to small changes in the mean velocity profile and to the exact form of the relation between $\mathscr{R}_{nm}$ and $\mathscr{U}_n$ than does the work done on the wave. This suggests that $\mathscr{U}_n$ may not be very similar under different experimental conditions and therefore does not provide a very useful test for theoretical predictions. Surface pressures alone do not provide a very stringent test on theories and it appears that the wave-induced mean velocity is not suitable; perhaps the variation with x_3 of averaged quantities like $\overline{\mathscr{U}_3\mathscr{P}}$ will prove most useful.

While measurements of mean flow properties such as $\mathscr{P}$ and $\mathscr{U}_n$ serve as tests of various possible models of turbulent stress generation they do not provide much help in developing better theories. Apparently what is required is very careful measurements of both $\mathscr{U}_n$ and the turbulent stresses $\mathscr{R}_{nm}$. These data can hopefully be used to formulate better constitutive relations; at least they can help answer such general questions as how directly related are $\mathscr{R}_{nm}$ and the local, rather than the global, structure of the mean flow.

Finally, the overall objective of this type of research deserves some comment. As our knowledge of airflow over waves and the related problem of wave generation increases it becomes increasingly clear that a single unified explanation will not be found immediately. Even if the airflow over small amplitude waves were understood, the generation of real waves would remain a difficult problem and it now appears that understanding the airflow involves acquiring a fairly complete picture of the generation of turbulent stresses. On the one hand, this means that solving the airflow problem will be very difficult but, as a result, it follows that this type of research is of greater scope than simply understanding the flow over waves; the problem is in fact an ideal context in which to develop and test theories of turbulent stress generation.

REFERENCES

BATCHELOR, G. K. 1967 *An Introduction to Fluid Dynamics*. Cambridge University Press.

BRADSHAW, P., FERRIS, D. H. & ATWELL, N. P. 1967 Calculation of boundary-layer development using the turbulent energy equation. *J. Fluid Mech.* **28**, 593.

DAVIS, R. E. 1969 On the high Reynolds number flow over a wavy boundary. *J. Fluid Mech.* **36**, 337.

DAVIS, R. E. 1970 On the turbulent flow over a wavy boundary. *J. Fluid Mech.* **42**, 721.

DOBSON, F. W. 1969 Observations of normal pressure on wind generated sea waves. Ph.D. dissertation, Department of Physics, University British Columbia.

HINZE, J. O. 1959 *Turbulence*. McGraw-Hill.

HUSSAIN, A. & REYNOLDS, W. C. 1970 The mechanics of a perturbation wave in turbulent shear flow. *Dept. of Mechanical Engineering, Stanford University Rep.* FM6.

KENDALL, J. M. 1970 The turbulent boundary layer over a wall with progressive surface waves. *J. Fluid Mech.* **41**, 259.

LIGHTHILL, M. J. 1962 Physical interpretation of the mathematical theory of wave generation by wind. *J. Fluid Mech.* **14**, 385.

LIN, C. C. 1955 *The Theory of Hydrodynamic Stability*. Cambridge University Press.

LUMLEY, J. L. & PANOFSKY, H. A. 1964 *The Structure of Atmospheric Turbulence*. Wiley.

MILES, J. W. 1957 On the generation of surface waves by shear flows. *J. Fluid Mech.* **3**, 185.

MILES, J. W. 1967 On the generation of surface waves by shear flows. Part 5. *J. Fluid Mech.* **30**, 163.

PHILLIPS, O. M. 1966 *The Dynamics of the Upper Ocean*. Cambridge University Press.

STEWART, R. H. 1970 Laboratory studies of the velocity field of deep-water waves. *J. Fluid Mech.* **42**, 733.

VOLKOV, IU. A. 1969 Spectra of velocity and temperature fluctuations in the airflow above the agitated sea surface. *Izv. Atmos. Oceanic Phys.* **5**, 1251.